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INTERVAL ANALYSIS: A NEW TOOL FOR APPLIED MATHEMATICS.(U)

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MRC Technical Summary Report # 2268

INTERVAL ANALYSIS: A NEW TOOL FOR
APPLIED MATHEMATICS

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August 1981

(Received July 15, 1981)

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INTERVAL ANALYSIS: A NEW TOOL FOR APPLIED MATHEMATICS

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ABSTRACT

Interval arithmetic has been found to be useful in numerical analysis as an automatic means to bound data, truncation, and roundoff errors in computations. Now that the speed of microprogrammed interval arithmetic approaches that of standard floating-point operations, a wider range of application to engineering and other problems has become feasible. Since, in many practical situations, data are only known to lie within intervals and only ranges of values are sought as satisfactory answers, straightforward interval computation can yield the desired results. Examples of this type of application are worst-case analysis of the stability of structures and the performance of electrical circuits. The recently developed theory of integration of interval functions also bears directly on the problems of solution of integral equations and the minimization of functionals defined in terms of integrals. Since certain chaotic phenomena, such as catastrophes and turbulence, are difficult to describe by single-valued functions, the introduction of interval functions and the corresponding analysis may lead to simpler models which will yield results of accuracy satisfactory for practical purposes.

(P1)

AMS (MOS) Subject Classifications: 28B20, 45L10, 65G10, 65K10, 65R20, 65V05

Key Words: Interval analysis, Interval integration, Interval iteration, Ranges of data and results, Error estimation, Analysis of structures and networks, Chaotic phenomena

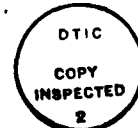
Work Unit Numbers 1 and 3 (Applied Analysis and Numerical Analysis and Computer Science)

SIGNIFICANCE AND EXPLANATION

This is a survey paper, written for applied mathematicians and engineers. The main idea is to present some of the basic principles and results of interval analysis in order to suggest applications, from straightforward engineering calculations to theoretical problems. For example, it is explained how interval analysis can be used to locate possibly troublesome parts of structures and electrical power networks, since responses to a range of loadings will be included in the results of an interval computation.

Interval analysis is presented as the branch of mathematics concerned with the study of the transformation of intervals into intervals, and thus is distinct from and complementary to real and complex analysis. Since this field is relatively new, and the ability to perform interval calculations at reasonable speed is just now becoming available on computers, a fertile area to explore seems to be to applications other than the well-established ones relating to error estimation in numerical analysis. Intervals and interval functions appear to be convenient means to describe the results of observations, which rarely yield exact real numbers and functions. In addition, many decisions must be made on the basis of a range of possible values of various parameters, and one is often interested in how a system will perform under a variety of conditions. Some physical phenomena, furthermore, are chaotic and difficult to describe by single-valued real or complex functions, an example being turbulent flow. Interval analysis provides a natural language for some problems of the types mentioned. Of course, before one can say whether or not interval methods are suitable for a given application, they must be tried. This paper points out possibilities, and provides a place from which to start in case interval techniques appear to be promising.

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INTERVAL ANALYSIS: A NEW TOOL FOR APPLIED MATHEMATICS

L. B. Rall

1. WHAT IS INTERVAL ANALYSIS ? This paper is addressed to the questions: What is interval analysis, and what can it do for applied mathematics? Naturally, these are big questions, so only a sketch of the answers can be given here. It is hoped, however, that enough can be conveyed to suggest possible useful applications of the subject, as well as to satisfy curiosity.

First of all, although interval analysis as a discipline is fairly recent (a likely beginning point is the Stanford Ph.D. thesis of R. E. Moore, written in 1962 [16]), it has grown beyond the scope of any single paper. Another survey has been given by Nickel [21], and there are at least three books on the subject [1], [17], [18], and the proceedings of three international conferences [9], [20], [22] are in print. A bibliography published in 1978 [2] lists 757 titles (this bibliography is reproduced in [18], pp. 125-179). Research in interval analysis and its applications is carried on most vigorously at the present time in the Federal Republic of Germany, where an Interval Library is maintained at the Institute for Applied Mathematics, University of Freiburg, under the supervision of Prof. Dr. Karl Nickel.

In spite of the extent of the subject, it is possible to characterize interval analysis by analogy with real analysis. In real analysis, the basic units are real numbers, and one studies transformations f of real numbers x into real numbers y , symbolized by

$$(1.1) \quad y = f(x).$$

Properties of these transformations (real functions) are of interest, as well

as operations (differentiation, integration, etc.) applied to these transformations.

Similarly, in interval analysis, the basic units are the nonempty closed intervals $X = [a,b]$ on the real line R , where

$$(1.2) \quad X = [a,b] = \{x \mid a \leq x \leq b, x \in R\}.$$

Interval analysis is thus concerned with transformations F of intervals X into intervals Y , that is

$$(1.3) \quad Y = F(X)$$

(see Figure 1).

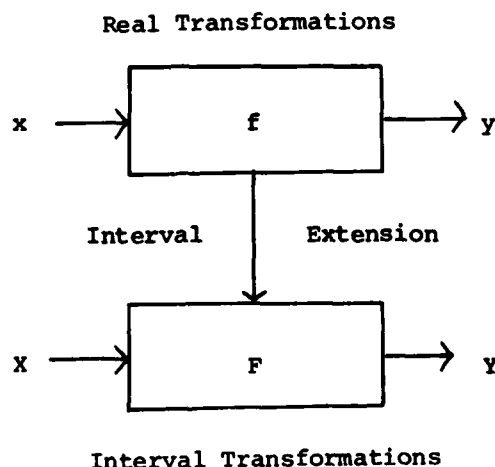


Figure 1. Relationship between Real and Interval Analysis

There is an obvious connection between interval analysis on the real line R and real analysis: One can identify each real number x with the corresponding degenerate interval $[x,x]$ having both endpoints equal to x , and write

$$(1.4) \quad x = [x,x].$$

Real and interval transformations are also related through the concept of interval extension, which will be treated in more detail in the next section.

The relationship between real and interval analysis is thus analogous to the relationship between real and complex analysis. The real numbers can also be identified with a subset of the complex numbers, and real transformations can be considered to be a restricted class of complex transformations. However, as everyone knows, complex analysis does not supersede real analysis, but is rather a complementary field of mathematics, with its own theory, techniques, and applications. Interval analysis also complements real analysis in a similar way.

Secondly, the question of the usefulness of interval analysis cannot be answered completely here, since one never knows entirely the capabilities of any mathematical theory or other tool, no matter how long its history. Here, a few examples of interval methods which have proved effective in practice will be given, and some speculation as to other applications will be made in hopes of stimulating further investigations into the uses of interval analysis in the solution of practical problems.

Before going on, it should be noted that just as real analysis extends from numbers R to real vectors in R^n , interval analysis also applies to interval vectors with n components. This simple generalization will be taken for granted below where appropriate. It should also be mentioned that there is a complex version of interval analysis, based on the use of disks or rectangles in the complex plane; attention here, however, will be restricted to real interval analysis.

From a philosophical point of view, it can be observed that measurements of physical phenomena do not yield real numbers and functions in general, but only approximations to these ideal concepts. However, it is usually possible to determine intervals in which the observed data lie, making interval analysis a natural language for the description of processes involving

inaccurately know data, or as a way to handle the results of variations in quantities of interest. Some illustrations of these ideas will be given below.

Before going on to a more detailed treatment of the subject, mention will be made of two problems which are solvable by the methods of interval analysis for which no techniques from real analysis are known:

(i) Global optimization: For $f: \mathbb{R}^n \rightarrow \mathbb{R}$, minimize f on \mathbb{R}^n (unconstrained optimization), or subject to the constraints

$$(1.5) \quad p_i(x) \leq 0, \quad i = 1, 2, \dots, m,$$

where f and p_1, \dots, p_m are at least once differentiable. Algorithms for the solution of these problems have been given respectively by Hansen [10] and Hansen and Sengupta [11].

(ii) Integration: In real analysis, theories of integration of real functions, such as those due to Riemann and Lebesgue [15] define the integral

$$(1.6) \quad I = \int_a^b f(x) dx$$

for only a subset of the real functions. In the theory of interval integration [6], all real functions (and all interval functions) are integrable. More details will be given in §§6-7 below.

2. INTERVAL EXTENSIONS. Figure 1 indicates another relationship between real and interval analysis, namely, interval extension of a real transformation. This concept is made precise in the following definition.

Definition 2.1. An interval transformation F is said to be an interval extension of a real transformation f if it has the following properties:

(i) inclusion,

$$(2.1) \quad f(X) = \{f(x) \mid x \in X\} \subset F(X);$$

and (ii) restriction,

$$(2.2) \quad F(x) = f(x),$$

where the convention (1.4) has been used to write $F(x) = F([x, x])$.

Of course, there are interval transformations F which are not extensions of real transformations f .

If f is a continuous function of a single variable, then $f(X)$ is an interval by the theorem of Weierstrass, and thus F defined by $F(X) = f(X)$ is an interval extension of f . However, in two or more dimensions, $f(X)$ is not in general an interval of the form $Y = ([c_1, d_1], \dots, [c_m, d_m])$ for $X = ([a_1, b_1], \dots, [a_n, b_n])$. For example,

$$(2.3) \quad f(x, y) = (x, y\sqrt{1-x^2})$$

maps $X_0 = ([0, 1], [0, 1])$ onto the region in the first quadrant bounded by the coordinate axes and a quarter of the unit circle, which is obviously not a rectangle. An interval extension of f given by (2.3) would have the property that $X_0 \subset F(X_0)$, since X_0 is the smallest rectangle X such that $f(X_0) \subset X$.

A fundamental interval extension of real transformations is interval arithmetic [17], [18], which extends the real arithmetic operations (considered as functions of two variables, that is, $f(x + y) = x + y$ for addition, etc.). The rules of interval arithmetic are as follows:

(i) Addition

$$(2.4) \quad [a, b] + [c, d] = [a + c, b + d].$$

(ii) Subtraction

$$(2.5) \quad [a, b] - [c, d] = [a - d, b - c].$$

(iii) Multiplication

$$(2.6) \quad [a, b] \cdot [c, d] = [\min \Pi, \max \Pi],$$

where $\Pi = \{ac, ad, bc, bd\}$.

(iv) Division

$$(2.7) \quad [a,b]/[c,d] = [a,b] \cdot [d^{-1},c^{-1}] \quad \text{if } cd > 0,$$

undefined otherwise.

These interval extensions of the ordinary arithmetic operations have the important property of inclusion monotonicity in the sense of the following definition.

Definition 2.2. An interval transformation F is said to be inclusion monotone if

$$(2.8) \quad X \subset Z \Rightarrow F(X) \subset F(Z).$$

Note that since interval analysis deals with set-valued quantities, then set relationships and operations, in certain instances, appear in a natural way.

The importance of interval arithmetic as defined by (2.4)-(2.7) is that it allows the construction of inclusion monotone interval extensions of rational functions automatically, simply by replacing the real variables by intervals and the arithmetic operations by their interval counterparts. For example,

$$(2.9) \quad F(X) = \frac{3 \cdot X + 1}{2 \cdot X - 1}$$

is an inclusion monotone interval extension of the real function

$$(2.10) \quad f(x) = \frac{3x + 1}{2x - 1}$$

on its domain of definition, which is the real line with the point $x = 1/2$ deleted.

Two cautions are in order concerning the straightforward use of interval arithmetic to obtain interval extensions. First of all, intervals do not form a linear space, as illustrated by the simple result,

$$(2.11) \quad [0,1] - [0,1] = [-1,1],$$

obtained from (2.5). Without a linear substructure, one cannot expect techniques from real analysis which depend on linearity to work in general in

interval analysis. For this reason, interval arithmetic cannot be applied indiscriminately to extend certain methods of linear algebra which are effective in the real case. In fact, the lack of a concept of linearity means that there is no way to distinguish between linear and nonlinear problems in interval analysis without going back to their real restrictions.

Secondly, the rules (2.4)-(2.7) of interval arithmetic are not adequate to produce small interval extensions of even some simple rational functions, for example, the extension

$$(2.12) \quad F(X) = X \cdot X$$

of

$$(2.13) \quad f(x) = x^2$$

gives

$$(2.14) \quad F([-1,1]) = [-1,1] \cdot [-1,1] = [-1,1],$$

while

$$(2.15) \quad f([-1,1]) = [-1,1]^2 = [0,1]$$

for the extension $f(X)$. Thus,

(v) Squaring

$$(2.16) \quad [a,b]^2 = \begin{cases} [\min\{a^2, b^2\}, \max\{a^2, b^2\}] & \text{if } ab > 0, \\ [0, \max\{a^2, b^2\}] & \text{if } ab < 0, \end{cases}$$

can be added to the rules for interval arithmetic to obtain improved interval extensions in the appropriate cases, and so on.

One way to improve the accuracy of interval extensions (in the sense of making $F(X)$ as small as possible) is thus to add additional operations to interval arithmetic. Another method is to use alternative expressions for the interval extension, and not just simple substitution of interval values and operations. For example, suppose that f is differentiable and f' has an interval extension F' . Furthermore, let $y = m(X)$ denote the midpoint of

$x; m([a,b]) = (a + b)/2$, with a similar expression for vector intervals X .

In this case, the mean value form [4]

$$(2.17) \quad F(X) = f(y) + F'(X) \cdot (X - y),$$

defines an inclusion monotone interval extension F of f which is accurate for small intervals X .

Further problems connected with the computation of interval extensions will be considered in the next section.

3. INTERVAL COMPUTATION. In actual practice, it is impossible in general to represent real numbers or evaluate real transformations exactly. This is because one must work with a finite set G of real numbers, called a grid [26] or screen [13]. A typical example of G is the set of fixed and floating-point numbers available on a given computer. The error introduced by having to work with G rather than R presents some thorny problems of numerical analysis in connection with the computation of real transformations. In interval analysis, on the other hand, the transition from R to G does not present any great theoretical difficulty, although one must forego the restriction property (2.2) of interval extensions in general. The construction of what will be called computable interval extensions having properties (2.1) and (2.8) of inclusion and inclusion monotonicity, respectively, will now be described.

It is helpful, but not necessary, to adjoin the extended real numbers $\pm \infty$ to G ; otherwise, as in actual practice, the calculation of numbers $x < \underline{g} = \min G$ or $x > \bar{g} = \max G$ is said to overflow the grid G , and will generate an error indication rather than a numerical result. With this in mind, attention will be restricted to the set RG of real numbers x such that $\underline{g} \leq x \leq \bar{g}$.

The set of intervals with endpoints in G will be denoted by IG , that

is,

$$(3.1) \quad IG = \{[a,b] \mid a,b \in G, a \leq b\}.$$

These are the intervals which are exactly representable using the available set of numbers. Now, the directed rounding operators Δ, ∇ from RG to G , and \otimes from IRG (the set of intervals with endpoints in RG) to IG will be defined.

For $x \in RG$, the upward rounding operator Δ is defined by

$$(3.2) \quad \Delta x = \min\{y \mid y \geq x, y \in G\},$$

and the downward rounding operator ∇ by

$$(3.3) \quad \nabla x = \max\{z \mid z \leq x, z \in G\}.$$

The directed rounding operator \otimes applied to $[a,b]$, where $a \leq b, a,b \in RG$ gives

$$(3.4) \quad \otimes[a,b] = [\nabla a, \Delta b],$$

an element of IG . In particular, if $x \in RG$ is a real number, then

$$(3.5) \quad \otimes x = \otimes[x,x] = [\nabla x, \Delta x],$$

using the identification (1.4) of real numbers with degenerate intervals.

For example, if G is the set of four digit decimal numbers, then \otimes applied to the real number $x = 1/3$ gives

$$(3.6) \quad \otimes(1/3) = [.3333, .3334],$$

which is the unique representation of $1/3$ in IG of minimal width (and also of each real number z satisfying $.3333 < z < .3334$).

It follows that \otimes is an inclusion monotone interval operator which maps IRG (and hence IG) into IG . If F is an interval extension of F , then

$$(3.7) \quad \Phi = \otimes F$$

will have properties (2.1) (inclusion) and (2.8) (inclusion monotonicity), and will map IG into IG . Interval operators Φ of this type will be called

computable interval extensions of f .

The advantages of working with computable interval extensions Φ of real transformations are obvious: If one starts with an exactly representable interval X (that is, $X \in IG$), then the transformed interval

$$(3.8) \quad Y = \Phi(X)$$

will also be exactly representable; furthermore, one can be sure that if $x \in X$, then $y = f(x) \in Y$. Much of the usefulness of interval computation stems from this latter fact.

On present computers, directed rounding and hence interval computation can be implemented by software [28] or microprogramming [19] with a certain amount of effort. Upward and downward rounding should, of course, be required to be available as a standard feature of future machines so that ordinary and interval arithmetic can be performed rapidly and accurately.

4. APPLICATIONS OF INTERVAL COMPUTATION. Interval computation, which here will mean calculation using computable interval extensions of real functions, has at least two closely related applications. The first, already widely used in numerical analysis, is to error estimation. It is assumed that there are exact data x and that one wishes the result $y = f(x)$ of performing an exact transformation f . However, all that one knows is that x lies in an interval X and that, due to inexact knowledge of coefficients, round-off and truncation error, the best one can compute is an interval transformation F of f . In this case, the interval transformation

$$(4.1) \quad Y = F(X)$$

is a precisely defined model of inaccurate computation on inexact data. Here, "the interval contains the answer," since $y \in Y$. Knowing this, it is easy to find an approximate value η of y and a corresponding bound ϵ for absolute, relative, percentage, or other error of η as an approximation to

y [23]. Most of the literature on interval analysis is devoted to applications of this type [2].

A second application of interval computation, possibly equally important, is to give a convenient, automatic way to estimate the effect of variation in input data on output results. For example, most structures, such as buildings, bridges, airframes, offshore drilling platforms, etc., are subject to a range of loadings, not just a single load, and the same is true for electrical power networks, pipelines, communications links, and so on. In these cases, the data X are really intervals, and one wants to examine an interval result Y to determine possible outcomes. In other words, "the interval is the answer" here. In prediction problems, also, the results of financial deals or the outcomes of battles may require estimation of interest rates, costs, military strengths and effectiveness, and so on. Interval analysis thus also provides a natural language for problems of this type, since one may view the results of assuming that the quantities of interest lie in various intervals, and make decisions accordingly.

To be more precise, consider the problem of the stability and safety of an offshore drilling platform attacked by waves having a range of amplitudes A and wavelengths λ , that is, by an interval wave. In real analysis, suppose that the deflection of the i -th joint (node) of the structure is given by a formula of the form

$$(4.2) \quad y_i = f_i(a, \lambda, x_1, \dots, x_m),$$

$i = 1, 2, \dots, n$, where the x_j represent loads, strengths of members (not usually known except within intervals), etc. Now, one calculates the interval values

$$(4.3) \quad Y_i = F_i(A, \lambda, X_1, \dots, X_m),$$

using suitable computable interval extensions. The result is essentially a

"worst-case" analysis, leading to the following conclusions:

(i) If all the Y_i lie within intervals of deflection considered to be acceptable, then the structure can be deemed to be safe under the given range of conditions.

(ii) On the other hand, if some of the computed intervals Y_i go beyond acceptable limits, this does not necessarily mean that the structure is unsafe. However, the nodes in question should be singled out for more exact investigation and possible reinforcement to insure that the structure will be stable.

It can be argued that the use of interval analysis is justified if this would prevent the failure of a single structure or power or communication network.

An interval analysis of linear electric circuits has been carried out by Skelboe [27] along the above lines. An application in finance, giving projected returns for interest rates lying in estimated intervals, has been worked out by A. S. Moore, and is given in [18], Chap. 9.

Many significant applications of the above idea are undoubtedly possible; the crucial point is to produce interval extensions which are realistic and accurate in the sense that the intervals obtained do not extend far beyond the limits which would actually be observed.

Another property of interval computation which is useful in applications is the intersection property. Suppose that one is trying to compute a point y , or, more generally, an interval Y , and one knows that

$$(4.4) \quad Y \subset Y_1$$

on the basis of an interval computation. Another computation gives, say,

$$(4.5) \quad Y \subset Y_2$$

It follows, since intervals are sets, that

$$(4.6) \quad Y \subset Y_1 \cap Y_2 = Y_3,$$

and Y_3 may be considerably smaller than Y_1 or Y_2 , but never larger. Thus, additional interval computations can only improve accuracy. This principle has been used in numerical integration to reduce error bounds, see [8] for an example.

Now that interval arithmetic can be microprogrammed to operate at essentially floating point speeds [19], and is available in the powerful scientific computing language PASCAL-SC [14], the use of interval computation in applications is in a position to grow at an explosive rate.

5. INTERVAL ITERATION. This section will deal with an application of interval computation to the problem of finding fixed points of real transformations, that is, real numbers, vectors, or functions y which satisfy the equation

$$(5.1) \quad y = f(y).$$

Since the formulation here is general enough to include systems of equations in several variables and integral equations [24], many problems in applied mathematics can be reduced to finding solutions of (5.1).

In real analysis, a standard way to obtain approximate solutions of equation (5.1) is by iteration. One starts with a point y_0 deemed to be a good approximation to y , and then generates the sequence $\{y_n\}$ by computing

$$(5.2) \quad y_{n+1} = f(y_n), \quad n = 0, 1, 2, \dots$$

If this sequence converges (and f is continuous, which will be assumed throughout), then

$$(5.3) \quad y = \lim_{n \rightarrow \infty} y_n$$

satisfies (5.1). In other words, convergence of $\{y_n\}$ implies the existence of a fixed point y for continuous f . This is all very well, but $\{y_n\}$ can diverge, even for y_0 close to y , and thus not yield any useful information (If there is no fixed point y of f , then $\{y_n\}$ must diverge, hence, nonexistence of y implies divergence of $\{y_n\}$.) Furthermore, since real

transformations cannot be performed exactly in general, one does not compute $\{y_n\}$, but rather some approximate sequence $\{z_n\}$. The usefulness of $\{z_n\}$ thus depends on a comparison with $\{y_n\}$ which can be a difficult problem in its own right (although this does provide employment for numerical analysts).

As shown in [26], the situation is quite different in interval analysis. Here, one starts with an interval Y_0 (in IG in actual practice) thought to contain a fixed point y of f , and computes the sequence of intervals $\{Y_n\}$ defined by

$$(5.4) \quad Y_{n+1} = Y_n \cap F(Y_n), \quad n = 0, 1, 2, \dots,$$

where F is a computable interval extension of f . In [26], it is shown that if $y \in Y_0$, then

$$(5.5) \quad y \in Y = \bigcap_{n=0}^{\infty} Y_n \neq \emptyset,$$

(\emptyset denotes the empty set), so existence of the fixed point y in Y_0 implies convergence of the interval iteration to a nonempty limit interval Y which also contains y . Furthermore, the endpoints of each Y_n , as well as Y , furnish lower and upper bounds for y . On the other hand, if

$$(5.6) \quad Y_N = \emptyset$$

for some positive integer N , then Y_0 contains no fixed points of y of f . In this case, the interval iteration (5.4) is said to diverge, which implies nonexistence of a fixed point in Y_0 .

For computations on a finite grid G , which is the case in actual practice, it has been shown [26] that

$$(5.7) \quad Y_N = Y_{N+1} = Y \text{ or } Y_N = \emptyset$$

for some positive integer N . Thus, interval iteration is a finite process in actual computation. In case $Y \neq \emptyset$, the usefulness of this limit can then be determined by direct inspection.

It should be noted that $Y \neq \emptyset$ does not imply that $y \in Y_0$; it is the

converse which holds. Thus, it may be necessary to apply some existence test to Y . However, in finite dimensions, it is sufficient that

$$(5.8) \quad F(Y_N) \subset Y_N$$

for some positive integer N to guarantee the existence of a fixed point $y \in Y_0$, and hence the convergence of the interval iteration as a consequence of the Schauder fixed point theorem. Interval iteration can thus be used to find improved bounds for fixed points known to be in Y_0 , to obtain regions Y_N possibly smaller than Y_0 to test for existence of fixed points, or to establish nonexistence of fixed points in Y_0 in case (5.6) holds ([18], Chapter 6).

6. INTERVAL FUNCTIONS. Interval functions are straightforward generalizations of real functions. Given real functions $y < \bar{y}$ (in the sense that $y(x) < \bar{y}(x)$ for each x), the function Y defined for each $x \in x = [a, b]$ by

$$(6.1) \quad Y(x) = \{y \mid y(x) < y < \bar{y}(x)\}$$

will be called an interval function on X with endpoint functions y, \bar{y} .

Figure 2 illustrates the graph

$$(6.2) \quad Y(X) = \{y \mid y(x) < y < \bar{y}(x), a < x < b\}$$

of a simple interval function.

As a set of functions, the interval function Y can be considered to be the set of all real functions y satisfying $y < y < \bar{y}$ in the sense cited above. Thus, in order to extend the concept of integration from real to interval functions, one must be prepared to integrate all real functions. The way to do this will be explained in the next section.

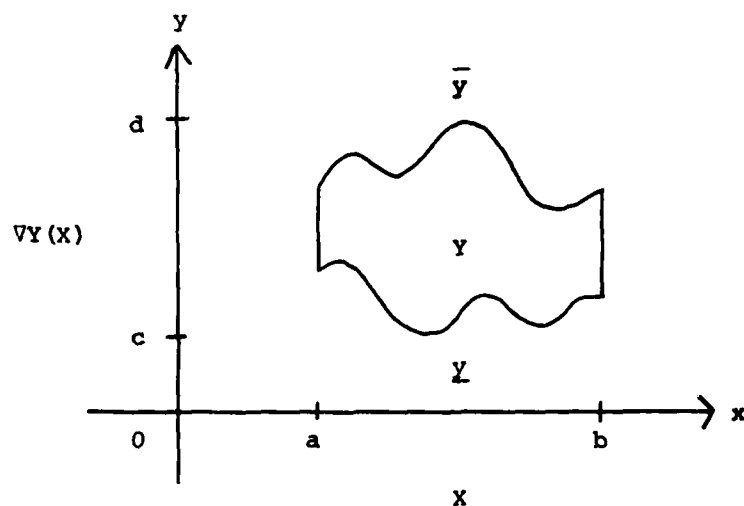


Figure 2. The Graph of an Interval Function.

An important interval function is the vertical extent VY of an interval function Y . This is defined by

$$(6.3) \quad VY(X) = [\inf_{x \in X} \{y(x)\}, \sup_{x \in X} \{\bar{y}(x)\}],$$

and is interval-valued. In Figure 2, $VY(X) = [c, d]$ is indicated on the y-axis. If $VY(X)$ is a finite interval, as in this case, then Y is said to be a bounded interval function.

Given interval functions Y, Z on X , one writes $Y \leq Z$ if $Y(X) \subseteq Z(X)$ as point-sets in the plane. Thus, if $Y = [y, \bar{y}]$ and $Z = [z, \bar{z}]$, then this is equivalent to $\underline{z} \leq \underline{y}$ and $\bar{y} \leq \bar{z}$ for the endpoint functions Y and Z . Given an interval function Y , it is possible on this basis to introduce the idea of directed rounding of Y to a larger interval function Z with endpoints belonging to a specified class (step-functions, splines, continuous or Riemann

integrable functions, etc.). This idea is useful in the construction of computable extensions of integral operators, etc.

Interval versions of discontinuous real functions can also be constructed to model finite or infinite jumps. For example, consider the real step-function

$$(6.4) \quad s(x) = \begin{cases} -1, & x < 1, \\ 1, & x > 1. \end{cases}$$

The corresponding interval step-function is

$$(6.5) \quad S(x) = \begin{cases} s(x), & x \neq 1, \\ [-1, 1], & x = 1, \end{cases}$$

as illustrated in Figure 3. Thus, interval step-functions include the "risers" as well as the "treads" of real step-functions, considered as mathematical models of staircases.

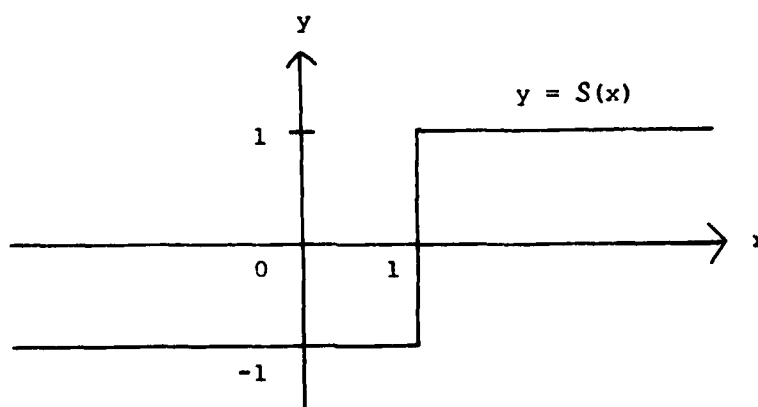


Figure 3. An Interval Step-Function.

Note that the graphs of interval versions of discontinuous real functions

will be connected sets in the plane. Interval functions of this type may be useful in the description of physical phenomena known as shocks or catastrophes, where rapid changes take place in certain quantities.

7. INTERVAL INTEGRATION. A construction of the interval integral and some of its important properties are given in [6]. Briefly, the idea is this: Let y be real function, and suppose \underline{S} and \bar{S} are the sets of step-functions \underline{s} , \bar{s} such that $\underline{s} < y$ for $\underline{s} \in \underline{S}$, and $y < \bar{s}$ for $\bar{s} \in \bar{S}$. Since step-functions are integrable in the extended real number system, one can always form the lower and upper Darboux integrals of y [15], denoted respectively by

$$(7.1) \quad (LD) \int_a^b y(x) dx = \sup_{\underline{s} \in \underline{S}} \left\{ \int_a^b \underline{s}(x) dx \right\},$$

and

$$(7.2) \quad (UD) \int_a^b y(x) dx = \inf_{\bar{s} \in \bar{S}} \left\{ \int_a^b \bar{s}(x) dx \right\}.$$

Definition 7.1. The interval

$$(7.3) \quad \int_a^b y(x) dx = [(LD) \int_a^b y(x) dx, (UD) \int_a^b y(x) dx]$$

is called the interval integral of the real function y over the interval $X = [a, b]$.

The interval integral, thus, always exists in the set of intervals on the extended real line [6] [15]. The interval integral (7.3) of a real function y is degenerate (a real number) if and only if y is Riemann (R) integrable, since

$$(7.4) \quad (R) \int_a^b y(x) dx = (LD) \int_a^b y(x) dx = (UD) \int_a^b y(x) dx$$

by definition of Riemann integration [15].

For Lebesgue (L) integrable functions y ,

$$(7.5) \quad (L) \int_a^b y(x) dx \in \int_a^b y(x) dx$$

in general [6].

Definition 7.2. The interval

$$(7.6) \quad \int_a^b Y(x) dx = [(\text{LD}) \int_a^b \underline{Y}(x) dx, (\text{UD}) \int_a^b \bar{Y}(x) dx]$$

is called the interval integral of the interval function Y over the interval $X = [a, b]$.

Of course, (7.6) reduces to (7.3) if $Y = [y, y]$ is degenerate (a real function). Interval integrals have many properties similar to those of real integrals, for example, the mean interval value theorem

$$(7.7) \quad \int_a^b Y(x) dx = w(X) \cdot \bar{Y}$$

holds, where $w(X) = W([a, b]) = b - a$, and \bar{Y} is some interval contained in $V_Y(X)$ [6].

Another important property of interval integrals is inclusion monotonicity, that is,

$$(7.8) \quad Y \subset Z \Rightarrow \int_a^b Y(x) dx \subset \int_a^b Z(x) dx,$$

as shown in [6]. Thus, interval integration is an inclusion monotone interval extension of real integration, with the restriction property (2.2) holding on the set of Riemann integrable functions by (7.4).

One useful application of interval integration is to construct computable interval extensions of integral transformations, which will be considered in the next section.

For bounded interval functions Y and finite intervals X , it has been shown that for the interval sums

$$(7.9) \quad \sum_n Y(X) = h \cdot \sum_{i=1}^n V_Y([a + (i-1)h, a + ih]), \quad h = \frac{b-a}{n}, \quad n = 1, 2, \dots,$$

one has

$$(7.10) \quad \int_a^b Y(x) dx = \bigcap_{n=1}^{\infty} \sum_n Y(X),$$

which gives a very simple construction of the interval integral in this case [25].

At the present time, differentiation does not seem to be an appropriate

concept for interval functions in general, except by means of interval extensions. Thus, if y is a smooth real function, then one may work with interval extensions Y of y , Y' of y' , Y'' of y'' , and so on, to obtain interval versions of results from real analysis. Indefinite interval integrals, however, have a "derivative" equal to their integrands at points of continuity of the latter (that is, points of continuity of both y and \bar{y}), just as in the case of real integrals [6].

8. APPLICATIONS OF INTERVAL INTEGRATION. As mentioned in §7, the theory of interval integration can be used to construct computable interval extensions of integral transformations of real functions, such as

$$(8.1) \quad Gy(x) = \int_a^b g(x, t, y(x), y(t)) dt.$$

In real analysis, one assumes Riemann or Lebesgue integrability of the integrand, which is not required in interval analysis. In any case, suppose that it is known that $y \in Y$, an interval function, and that Y is a computable interval extension of g obtained by interval arithmetic or otherwise, in the sense that the endpoint functions \underline{Y}, \bar{Y} of Y have computable Riemann integrals. (It may be necessary to use directed rounding to obtain \underline{Y}, \bar{Y} .) Then, Γ defined by

$$(8.2) \quad \Gamma Y(x) = \int_a^b Y(x, t, Y(x), Y(t)) dt$$

will be a computable interval extension of the integral operator G .

Two applications of this idea will be mentioned: The solution of integral equations and the minimization of functionals. First, consider the fixed point problem for G defined by (8.1) with $X = [0, 1]$, that is, the integral equation

$$(8.3) \quad y(x) = \int_0^1 g(x, t, y(x), y(t)) dt.$$

Equation (8.3) is very general. It is of Volterra type if

$g(x,t,u,v) \equiv 0$ for $t > x$ (or more generally, for $t > h(x)$), in which case the upper limit of integration in (8.3) can be replaced by x (or $h(x)$); otherwise, (8.3) is of Fredholm type. Linear integral equations of first, second, and third kinds correspond to the following integrands:

$$\begin{aligned} \text{1st kind: } & g(x,t,y(x),y(t)) = y(x) + f(x) + \lambda K(x,t)y(t), \\ (8.4) \quad \text{2nd kind: } & g(x,t,y(x),y(t)) = f(x) + \lambda K(x,t)y(t), \\ \text{3rd kind: } & g(x,t,y(x),y(t)) = (1 - \phi(x))y(x) + f(x) + \lambda K(x,t)y(t). \end{aligned}$$

Among nonlinear integral equations of the form (8.3) which are of special interest are the equations of Hammerstein type,

$$(8.5) \quad g(x,t,y(x),y(t)) = K(x,t)f(t,y(t)),$$

Urysohn type,

$$(8.6) \quad g(x,t,y(x),y(t)) = f(x,t,y(t)),$$

the Chandrasekhar H-equation [7],

$$(8.7) \quad g(x,t,y(x),y(t)) = 1 + \lambda y(x) \frac{t\psi(t)}{x+t} y(t),$$

and many others.

An obvious approach to the approximate solution of equation (8.3) is the use of interval iteration,

$$(8.8) \quad Y_{n+1} = \Gamma Y_n \cap Y_n, \quad n = 0, 1, 2, \dots,$$

starting with an interval function Y_0 which is presumed to contain a solution y of the integral equation. The theory of interval iteration [24], [26] applies in this case also: If $y \in Y_0$, then (8.8) will converge to an interval function $Y = [\underline{y}, \bar{y}]$ such that

$$(8.9) \quad \underline{y}(x) \leq y(x) \leq \bar{y}(x), \quad 0 \leq x \leq 1,$$

thus giving pointwise bounds for the solution y of the integral equation

(8.3). On the other hand, if $Y_N = \emptyset$ for some positive integer N , in the sense that $Y_{N-1}(x) \cap \Gamma Y_{N-1}(x) = \emptyset$ for some x , so that Y_N is not defined as an interval function, then there is no solution of (8.3) such that

$$(8.10) \quad \underline{y}_0(x) \leq y(x) \leq \bar{y}_0(x), \quad 0 \leq x \leq 1,$$

that is, the interval function $Y_0 = [y_0, \bar{y}_0]$ does not contain a fixed point y of G [24].

In the case in which the limit of the interval iteration (8.8) is degenerate, the following regularity theorem holds: If the interval iteration (8.8) converges and

$$(8.11) \quad \lim_{n \rightarrow \infty} \Gamma Y_n = y,$$

a real function, then y satisfies (8.3) in the sense of Riemann (R) integration [24], that is,

$$(8.12) \quad y(x) = (R) \int_0^1 g(x, t, y(x), y(t)) dt.$$

This kind of convergence will occur if the interval operator Γ is what is called an interval contraction [3], [5]. Even if G and hence Γ are not contractive operators, interval iteration can be used to obtain improved lower and upper bounds for y [24].

Another application of interval integration is to the minimization of functionals, to which many problems in applied mathematics reduce. For example, instead of the functional.

$$(8.13) \quad f(y) = \int_a^b \phi(x, y(x), y'(x)) dx,$$

one can examine the interval functional

$$(8.14) \quad F(Y) = \int_a^b \Phi(x, Y(x), Y'(x)) dx,$$

where Φ, Y, Y' denote computable interval extensions of ϕ, y, y' ,

respectively. It follows that (8.14) provides immediate lower and upper bounds for the values of the functional (8.13) for $y \in Y, y' \in Y'$, that is, if $F(Y) = [c, d]$, then $c \leq f(y) \leq d$ for $y \in Y, y' \in Y'$. An algorithm similar to the one of Hansen and Sengupta [11] could then be applied to locate and obtain lower and upper bounds for

$$(8.15) \quad \min f(y), y \in Y, y' \in Y'.$$

Since many physical principles, ordinarily formulated as differential equations, have alternative formulations as minima, maxima, or stationary

values of functions expressed in integral form, or as integral equations, this is an area in which interval analysis can be extremely useful, particularly if the data are inexactly or incompletely known. This is a topic for future research; others will be indicated in the next section.

9. DIRECTIONS FOR FUTURE RESEARCH. Research in applied mathematics follows two closely related lines: Application of known mathematical techniques to problems of importance in practice, and the development of new methods when known ones are inadequate or inefficient. The applied mathematician thus functions both as problem-solver and as "toolmaker to the trade". To a certain extent, the emphasis on tool making to occurs in academic environments, and on using tools in laboratories. The U. S. Army Mathematics Steering Committee and the U. S. Army Research Office have provided a valuable service for many years by organizing meetings such as the Conferences of Army Mathematicians, which bring together applied mathematicians with both theoretical and practical orientations. To these groups, interval analysis is hereby offered as a new tool. It will work well on some problems, not on others, and will need improvement to be effective in other cases.

The usefulness of interval computation as described in §4 is well-established by now. Given the increased availability of fast interval arithmetic, interval analysis can and will be applied to more computational problems of the type described. This also applies to the solution of systems of equations (§5), integral equations (§8), and finding lower and upper bounds for values of functionals, as described also in §8.

In a more speculative vein, it appears that interval functions might provide a more realistic description of chaotic phenomena, such as turbulent flow, than single-valued real or complex functions. Also, since many physical principles have integral as well as differential formulations, interval integration might be applicable to a whole range of problems now solved

approximately by the numerical integration of ordinary or partial differential equations. While interval analysis may or may not work in some of these areas, it has enough potential to at least be investigated, which is what research in applied mathematics is all about.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TSR#2268	2. GOVT ACCESSION NO. AD-A110417	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Interval Analysis: A New Tool for Applied Mathematics		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) L. B. Rall		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 (Applied Analysis) 3 (Numerical Analysis and Computer Science)
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE August 1981
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 26
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Interval analysis, Interval integration, Interval iteration, Ranges of data and results, Error estimation, Analysis of structures and networks, Chaotic phenomena		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Interval arithmetic has been found to be useful in numerical analysis as an automatic means to bound data, truncation, and roundoff errors in computations. Now that the speed of microprogrammed interval arithmetic approaches that of standard floating-point operations, a wider range of application to engineering and other problems has become feasible. Since, in many practical situations, data are only known to lie within intervals and		

20. ABSTRACT, con't.

only ranges of values are sought as satisfactory answers, straightforward interval computation can yield the desired results. Examples of this type of application are worst-case analysis of the stability of structures and the performance of electrical circuits. The recently developed theory of integration of interval functions also bears directly on the problems of solution of integral equations and the minimization of functionals defined in terms of integrals. Since certain chaotic phenomena, such as catastrophes and turbulence, are difficult to describe by single-valued functions, the introduction of interval functions and the corresponding analysis may lead to simpler models which will yield results of accuracy satisfactory for practical purposes.